Sequential Formation of Alliances in Survival Contests

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January 31, 2019

Abstract

We consider a sequential formation of alliances à la Bloch (1996) and Okada (1996) followed by a two-stage contest in which alliances …rst compete with each other, and then the members in the winning alliance compete again for an indivisible prize. In contrast to Konishi and Pan (2019) which adopted an open-membership game as the alliance formation process, alliances are allowed to limit their memberships (excludable alliances). We show that if members' e¤orts are strongly complementary to each other, there will be exactly two asymmetric alliances— the larger alliance is formed …rst and then the rest of the players form the smaller one. This result contrasts with the one under open membership, where moderate complementarity is necessary to support a two-alliance structure. It is also in stark contrast with Bloch et al. (2006), where they show that a grand coalition is formed in the same game if the prize is divisible and a binding contract is possible to avoid further con‡icts after an alliance wins the prize.

1 Introduction

In their in‡uential paper, Esteban and Sákovics (2003) consider a three-person strategic alliance formation in a Tullock contest model in which players compete for an indivisible prize, and demonstrate that an alliance involves strategic disadvantages (see also Konrad 2009). There are two main disadvantageous forces against forming an alliance: First, if an alliance is formed, there will be

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and end up with a trivial grand alliance.[4] They show that for intermediate values of the CES complementarity parameter, there exists a unique nontrivial two-alliance equilibrium.

In contrast, in this paper, we use Bloch's (1996) and Okada's (1996) sequential coalition (alliance) formation game (along the line of a noncooperative coalition bargaining game in Chatterjee, et al. 1993). Although the open-membership game in Konishi and Pan (2019) is widely used in coalition formation games, the non-excludability – that is, players are allowed to freely complementarity parameter under a small number of players (ten players). We show that there will be no alliance if   is small, but as   goes up the sizes of alliances increase. Once   passes a certain threshold value, there will be only two (asymmetric) alliances in equilibrium, and every player participates in alliances as we have shown in our main theorem.

The rest of the paper is organized as follows. In the next subsection, we review the relevant literature. Section 2 introduces the model, and Sections 3 and 4 investigate subgames in stages 3 and 2, respectively. Section 5 presents results on equilibrium alliance structures, and Section 6 provides numerical examples. Section 7 concludes.

1.1 Literature Review

Since we provide a general literature review in our companion paper (Konishi and Pan 2019), we will concentrate on the games that determine an alliance structure. In the companion paper, we used so-called open-membership game where all players can move freely without being excluded from alliances.[5] However, depending on the nature of alliances we consider, we may want to see how equilibrium alliance structure is affected by allowing exclusive memberships of alliances.

Although we can think of different ways to introduce "excludability" of alliance memberships in an alliance formation game (see Hart and Kurz 1983, and Bloch 1997), the most popular way in the literature is to extend Rubinstein's two-person noncooperative bargaining game to a sequential coalition formation game: Chatterjee et al. (1993), Bloch (1996), Okada (1996), and Ray and Vohra (1999), among others. Although their games differ in the methods of choosing the proposers (following different protocols), the procedures for forming coalitions are the same. At each stage, a proposer proposes a coalition she belongs to, and ask the members of the coalition whether or not they accept the offer. If every member accepts the offer, then the coalition is formed, and the leftover players continue to form coalitions by the same procedure. If any of the members of a proposed coalition rejects the offer, the coalition is not formed, and a new proposer is specified by the protocol.

In the context of contests, Bloch et al. (2006) generalize the model substantially to analyze the stability of the grand alliance in different alliance formation games, including a sequential coalition formation game in Bloch (1996). Sánchez-Páges (2007a) explores different types of stability concepts

[5] Baik and Lee (1997, 2001) use open-membership games to describe alliance formation in endogenizing the alliance structure in Nitzan's (1991) game with endogenous group sharing rules.

including sequential coalition formation games in alliance formation in con-and a time discount factor $\in (0,1)$ applies to the final payoff. The process continues until there is no player left and $= \{S_1, S_2, \ldots, S_J\}$ is formed.

We introduce potential benefits for players who belong to an alliance—complementarity in aggregating efforts by all alliance members. That is, if player i belongs to alliance $j = 1, \ldots, J$ with $S_j \subset N$ as the set of members, and these members make efforts $(e_{hj})_{h \in S}$, then the aggregated effort of alliance j, E_j, is described by a CES aggregator function

$$E_j = \left(\sum_{h \in S} e_{hj}^{1-} \right)^{\frac{1}{1-}}; \tag{1}$$

where $\in (0,1]$ is a parameter that describes the degree of complementarity: if $= 0$ it is a linear function, and if $= 1$ it is a Cobb-Douglas function. Thus, as goes up, the complementarity of members' efforts increases.

Candidate i in alliance j decides how much effort e_{ij} to contribute to her alliance j. The winning probabilities of an alliance is a Tullock-style contest. That is, an alliance j's "winning probability" given its members' efforts is

$$p_j = \frac{E_j}{\sum_{k \in J} E_k}. \tag{2}$$

An indivisible prize is valued as $V > 0$, which is common to all players. Since the prize is indivisible, one player in the winning alliance in the second stage must be selected as the final winner in the third-stage contest.

In the third-stage competition, we assume that a Tullock contest takes place within the winning alliance S_j. Denoting the second-stage effort as $\hat{e}_i$, the winning probability of player $i \in S_j$ is

$$p_i = \frac{\hat{e}_i}{\sum_{h \in S} \hat{e}_h} \tag{3}$$

Formally, an alliance structure is a partition of the set of players N, $= \{S_1, \ldots, S_J\}$, where each alliance j consists of a set of players S_j and $\cup_{j \in J} S_j = N$, and $S_{j'} \cap S_j = \emptyset$ for any $j, j' \in \{1, \ldots, J\}$ with $j \neq j'$. Since we assume that players are ex-ante homogenous, we also call $\{n_1, \ldots, n_J\}$ an alliance structure with $n_j = |S_j|$ for all $j = 1, \ldots, J$. Our three-stage dynamic contest game with sequential alliance formation is summarized as:

Stage 1. In round $j = 1, 2, \ldots,$ one player is selected as a proposer with equal probability among all active players in the round j, N_j, where $N_1 = N$.[7]

[7] This is the random proposer protocol put forth by Okada (1996). Bloch (1996) uses a deterministic protocol, but the results we obtain in these two setups are the same if effort complementarity is high enough.

The selected player proposes an alliance $S_j \subseteq N_j$. All other players in S_j either accept or reject the proposal sequentially. If all other players in the alliance S_j accept the proposal, S_j is formed and removed from the process, and $j+1$ round starts with the remaining players $N_{j+1} = N_j \backslash S_j$. Otherwise, payoff discounts by $\delta \in (0;1)$ apply to all players, the round $r+1$ starts with $N_{j+1} = N_j$ by the same rule. The process continues until there is no player left and $\mathcal{S} = \{S_1; S_2; ::::; S_J\}$ is formed.[8]

Stage 2. All players $i \in N$ choose effort $e_i \in \mathbb{R}_+$ simultaneously, knowing the aggregated effort of her alliance is (1). The inter-alliance contest is a Tullock contest with winning probabilities equal to (2).

Stage 3. All members of the winning alliance S_j choose effort $\hat{e}_i \in \mathbb{R}_+$ simultaneously. The ultimate winner is selected by a simple Tullock contest with winning probabilities equal to (3).

We use standard subgame perfect Nash equilibrium as the solution of this dynamic game. We consider equilibria in pure strategies only. We will analyze this game by backward induction.

3 Equilibrium

3.1 Stage 3: Final Contest within the Winning Alliance

In the third stage, all members in the winning alliance S_j in the first stage engage in a Tullock contest by exerting effort $\hat{e}_i \geq 0$. Thus, player i's winning probability is

$$p_i = \frac{\hat{e}_i}{\sum_{h \in N} \hat{e}_h}.$$

For any player i in the winning group j, the expect payoff in stage 3 is

$$\hat{V}_i =$$

Since players are homogeneous, $p_i(1 - p_i) = \frac{n-1}{n^2}$ is the same for all i in the winning group j. Then, we have the following proposition.

Proposition 1. Suppose that the winning alliance of the first stage has size n_j. Then, the second-stage equilibrium strategy and payoff are

$$\hat{e}^j = \frac{n_j - 1}{n_j^2} V \text{ and } \hat{\pi}^j = \frac{V}{n_j}\left(1 - \frac{n_j - 1}{n_j}\right) = \frac{V}{n_j^2}$$

3.2 Stage 2: Contest between Alliances

Consider an inter-alliance contest problem. Without loss of generality, we reorder any alliance structure Π from the first stage so that $n_1 \geq n_2 \geq \ldots \geq n_{J^*}$. From Proposition 1, we know that for a given size of alliance n_j the payoff of intra-alliance contest is determined by $\hat{\pi}_j = \frac{V}{n^2}$. In the companion paper, Konishi and Pan (2019) have the following result.

Theorem 1. (Konishi and Pan, 2019) There exists a unique equilibrium in the second stage for any partition of players $\Pi = \{n_1, \ldots, n_{J^*}\}$ characterized by $\bar{j} \in \{1, \ldots, J^*\}$ such that $p_j > 0$ (active alliance) for all $j \leq \bar{j}$, while $p_j = 0$ (inactive alliance) for all $j > \bar{j}$. Moreover, the members of alliance $j = 1, \ldots, J^*$ obtain payoff

$$u_j = \begin{cases} \frac{1}{n^2}\left[1 - (\bar{j} - 1)\frac{n^{\frac{2-3\alpha}{1-\alpha}}}{\sum_{\ell=1}^{\bar{j}} n_\ell^{\frac{2-3\alpha}{1-\alpha}}}\right. \end{cases}$$

3.3 Stage 1: Alliance Structures under Sequential Coalition Formation

Here, we consider a sequential coalition formation game with exclusive alliances a la Bloch (1996) and Okada (1996). The main results are as follows.

Theorem 2. For any N, there is (N) such that, for all $\geq (N)$, there are only two alliances in equilibrium. All players belong to one of the two and

$$g(x;x;J+1) = \frac{J\frac{1}{x} - (J-1)\frac{1}{x}}{x^2 \; J\frac{1}{x} + \frac{1}{x}}$$

We have the following result.

Lemma 1. Suppose that $J \geq 1$ alliances with their average size x have been formed and remain active even with the entry of the $J+1$ alliance. Then, (i) $\frac{@u(x;x;J+1)}{@x} < 0$ for all x and x, and (ii) $\frac{@u(x;x;J+1)}{@x} > 0$ holds for all $\frac{J \;\; 1}{J}x \leq \frac{x}{x} \;\; \leq \frac{(2+\;\;)J \;\; 4}{2J}$ when $\;\; \geq 2$. Moreover, if $\frac{x}{x} \;\; < \frac{J \;\; 1}{J}$, then even if the $J+1$th alliance with size x enters, it cannot be active.

The implications of this lemma are listed in the following corollaries.

Corollary 1. When $\;\; > \frac{4}{J}$, then the best response of the $J+1$th alliance satis…es $x > x$ knowing that the alliance with a higher winning probability dominates the loss from sharing with a larger group.

Lemma 6. Suppose that among J formed alliances, $J^M \geq 1$ of them have the largest size x^M, and $x^M < 1 - \sum_{j=1}^{J} x_j < 2x^M$. For $\geq \tilde{}(N)$ for some $\tilde{}(N)$, we have $u(x^M + \frac{1}{N}; x^M; J^M) > u(x; x^M; J^M)$ for all $x \leq x^M$, and $u(x^M; x^M + \frac{1}{N}; J^M) < u(\frac{1}{2}\ 1 - \sum_{j=1}^{J} x_j\ + \frac{1}{N}; \frac{1}{2}\ 1 - \sum_{j=1}^{J} x_j\ - \frac{1}{N}; 1)$. That is, the benefits of belonging to a larger alliance with a higher winning probability dominates the losses of sharing with a larger group.

Proof of Theorem 2. We can rename (N) by the maximum of the original (N), $\hat{}(N)$, (N), and $\tilde{}(N)$. Let (N) be that corresponds to (N): By the sequence of the lemmas above, we consider the second mover's best or better responses.

1. Suppose that $x_1 \geq \frac{1}{2}$. By Lemma 1, $x_2 = 1 - x_1$ is the best response.
2. Suppose that $\frac{1}{3} \leq x_1 < \frac{1}{2}$. Suppose that $x_2 \leq \frac{1\ \ x_1}{2}$. We will show that forming multiple same-size alliances is dominated by forming an alliance of size $x_1 + \frac{1}{N}$. Suppose that two or more size-x_2 alliances are formed after a size-x_1 alliance. In this case, $x_2 \leq x_1$ holds. By Lemma 3, having only one size-x_2 alliance is generally better than forming multiple of them. Since $x_2 \leq x_1$, calling x_2 is dominated by calling x_1 by Lemma 1. But Lemma 4 suggests that for the second mover calling $x_1 + \frac{1}{N}$ dominates calling x_1, since Lemma 2 implies that there will be only two active alliances if $x_1 +$ is called. NNNNI[]0d0J0.478+

this behavior by the $J-1$th alliance, the $J-2$th alliance can call a little more than one half of the set of players who do not belong to alliances 1 to $J-3$ (Lemma 6). Then, only the $J-2$th and the $J-1$th alliances will remain active, and alliance 1 gets zero payoff (the $J-1$th alliance is formed by all of the rest of the players by Lemma 1). Thus, this case cannot be an equilibrium as well.

Hence, only case 1 can happen in equilibrium, and there are only two alliances in equilibrium, all players belong to one of the alliances, and the first alliance is larger than the second.

Remark. Since $x_1 > x_2$ holds with $u(x_1; x_2; 2) > u(x_2; x_1; 2)$ in equilibrium, there will not be any delay in forming coalitions. That is, the same outcome would realize independent of the protocol.

4 Examples with Small Population

For our analytical result, we will consider the cases of relatively low complementarity parameter with a small number of players $N = 10$. The complementarity parameter value $\geq \frac{6}{}$

(0:027344; 0; 027344). If the second alliance calls a size 3 alliance, then the third alliance will be size 3, and their payoffs for (4; 3; 3) are (0:04 2323; 0:010809; 0:010809). Thus, the second mover will call a size 5, and the payoffs for (4; 5) are (0012521; 0:028641).

5. The first mover calls a size 3 alliance. If the second mover calls a size 3, then the rest form a size 4, and this is not beneficial for the second mover (see above). If she calls a size 4, then (3; 4; 3) realizes with (0:01 0809; 0:042323; 0:010809). If she calls a size 5, then (3; 5) realizes, leaving an inactive size 2 alliance with payoffs (0:0089861; 0:034598). So, her best response is to call a size 4 alliance.

6. The first mover calls a size 2 alliance. Then, the second mover calls a size 5 alliance, making the first mover's alliance inactive. The payoffs for (5; 3) are (0:034598; 0:0089861).

In summary, the first mover calls size 6 alliance. The first two alliances' payoffs from (6; 4) are (0:022558; 0:0081571).

4.2 Case 2: $= \frac{5}{6}$ or $= 3$

When $= \frac{5}{6}$, the general pattern is similar to the case of $= \frac{6}{7}$, except for

4.3 Case 3: Smaller s

When $= \frac{4}{5}$ ($= 2$), the situation is the same as in the $= \frac{5}{6}$ case. The equilibrium (active) alliance structure for this case is $(4; 4)$. How about for an even smaller ? When $= \frac{3}{4}$ ($= 1$), we have an (active) equilibrium alliance structure $(3; 3; 3)$, achieving payo¤s $0{:}028807$. Note that this number is higher than the payo¤ from $(4; 4)$, $0{:}027344$. With this low complementarity, even if the …rst mover calls a size 3 alliance, the second mover does not bene…t by calling a size 4 or 5 alliance. Having a large alliance just intensi…es the subsequent …ght, and $(3; 3; 3)$ realizes.

When $= \frac{2}{3}$ ($= 0$), the equilibrium alliance structure is $(2; 2; 2; 2; 2)$ with payo¤s $0{:}03$. There will be no further spino¤ for this $N = 10$, since calling a one person alliance increases the number of alliances, which is harmful to the player (an independent player gets $\frac{1}{36} < 0{:}03$ from $(2; 2; 2; 2; 1; 1)$). However, if N goes up, all alliances are resolved, going back to the standard Tullock competition.

5 Concluding Remarks

In this paper, we consider an alliance formation game in Tullock contests when e¤orts by the members of an alliance are complementary to each other. In order to illustrate excludability of alliance memberships, we use Bloch's noncooperative game of sequential coalition formation (1996). Unlike in an open-membership game analyzed in the companion paper (Konishi and Pan 2019), strong complementarity does not mean a grand alliance, since alliances can exclude outsiders by limiting membership. We show that there will be only two asymmetric alliances in which (i) all players belong to one of them, and (ii) the …rst alliance is larger than the second alliance, when e¤ort complementarity is large enough. With a small population example, we show that (i) there can be more than two alliances in equilibrium, and (ii) there can be fringe inactive players in equilibrium when e¤ort complementarity is not too strong. These results sheds light on the role of exclusivity in forming alliances in the context of contest games.

Appendix

We collect all the proofs of lemmas in the text.

Proof of Lemma 1. We start by di¤erentiating f and g with respect to x:

$$\frac{@f(x; x; J+1)}{@x} = \frac{-\ J \frac{1}{x\ +1} J \frac{1}{x\ +1}}{J\frac{1}{x} + \frac{1}{x}\ ^2} < 0$$

and

$$\begin{aligned}
\frac{@g(x; x; J+1)}{@x} &= \frac{-\ J\frac{1}{x\ +1}\ \ J\frac{1}{x} + \frac{1}{x}\ \ +\ J\frac{1}{x} - (J-1)\frac{1}{x}\ \ \ J\frac{1}{x\ +1}}{x^2\ J\frac{1}{x} + \frac{1}{x}\ ^2} \\
&= \frac{-\ J\frac{1}{x\ +1}\ \ J\frac{1}{x} + \frac{1}{x}\ -\ J\frac{1}{x} - (J-1)\frac{1}{x}}{x^2\ J\frac{1}{x} + \frac{1}{x}\ ^2} \\
&= \frac{-\ J\frac{1}{x\ +1} \times J\frac{1}{x}}{x^2\ J\frac{1}{x} + \frac{1}{x}\ ^2} < 0
\end{aligned}$$

These imply that $\frac{@u(x;x;J+1)}{@x} < 0$: i.e., a coalition's payo¤ declines if other active coalitions' sizes increase.

Di¤erentiating f and g with respect to x, we have

$$\begin{aligned}
\frac{@f(x; x; J+1)}{@x} &= \frac{J}{N} \frac{(\ \ +1)\frac{1}{x\ +2}\ \ J\frac{1}{x} + \frac{1}{x}\ \ + \frac{1}{x\ +1}\ -\ \frac{1}{x\ +1}}{J\frac{1}{x} + \frac{1}{x}\ ^2} \\
&= \frac{J}{N} \frac{(\ \ +1)\, J\frac{1}{x\ +2}\frac{1}{x} + \frac{1}{x^2\ +2}}{J\frac{1}{x} + \frac{1}{x}\ ^2} > 0
\end{aligned}$$

$$\begin{aligned}
\frac{@g(x; x; J+1)}{@x} &= \frac{(J-1)\frac{}{x\ +1}x^2\ \ J\frac{1}{x} + \frac{1}{x}\ -\ J\frac{1}{x} - (J-1)\frac{1}{x}\ \ \ 2x\ \ J\frac{1}{x} + \frac{1}{x}\ - x^2\frac{}{x\ +1}}{x^4\ \ J\frac{1}{x} + \frac{1}{x}\ ^2} \\
&= \frac{(J-1)\frac{}{x\ +1}x\ \ J\frac{1}{x} + \frac{1}{x}\ -\ J\frac{1}{x} - (J-1)\frac{1}{x}\ \ \ 2\ \ J\frac{1}{x} + \frac{1}{x}\ - x\frac{}{x\ +1}}{x^3\ \ J\frac{1}{x} + \frac{1}{x}\ ^2} \\
&= \frac{(J-1)\frac{}{x} - 2\ \ J\frac{1}{x} - (J-1)\frac{1}{x}\ \ \ J\frac{1}{x} + \frac{1}{x}\ +\ J\frac{1}{x} - (J-1)\frac{1}{x}\ \ \frac{}{x}}{x^3\ \ J\frac{1}{x} + \frac{1}{x}\ ^2} \\
&= \frac{(J-1)(\ \ +2)\frac{1}{x} - 2J\frac{1}{x}\ \ \ J\frac{1}{x} + \frac{1}{x}\ +\ J\frac{1}{x} - (J-1)\frac{1}{x}\ \ \frac{}{x}}{x^3\ \ J\frac{1}{x} + \frac{1}{x}\ ^2} \qquad (4)
\end{aligned}$$

Thus, $\frac{@g(x;x;J+1)}{@x} > 0$ (thus $\frac{@u(x;x;J+1)}{@x} > 0$) holds if we have

$$\frac{J-1}{J} \leq \frac{x}{x} \leq \frac{(2+\ \ )(J-1)}{2J}$$

We can relax the su¢ cient upperbound slightly:

respectively. We have

$$Jx_2$$

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