# Real Analysis Qualifying Exam

*Answer all four questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. All four questions are worth the same number of points. Good luck.*

**Notation:** In the questions below, $\lambda^n$ denotes Lebesgue measure on $\mathbb{R}^n$.

**Question 1.** Let $\{f_1, f_2, \ldots\}$ be a sequence of continuous, positive functions defined on the unit interval $[0, 1]$ with

$$\int_0^1 f_n(x)\, d\lambda^1(x) = 1$$

for all $n$. Assume that the pointwise limit of the sequence $\{f_n\}$ exists, and denote it by $f$.

a. Is it always true that $\int_0^1 f(x)\, d\lambda^1(x) \leq 1$? Prove or provide a counterexample.

b. Is it always true that $\int_0^1 f(x)\, d\lambda^1(x) \geq 1$? Prove or provide a counterexample.

**Question 2.** For any $\lambda^2$-measurable function $f : \mathbb{R}^2 \to \mathbb{R}$, and for every $x, y \in \mathbb{R}$, define $f_x : \mathbb{R} \to \mathbb{R}$ and $f^y : \mathbb{R} \to \mathbb{R}$ by $f_x(p) = f(x, p)$ and $f^y(p) = f(p, y)$.

a. Given an example of such a function $f$ such that $f_x \in L^1(\mathbb{R})$ for a.e. $x$ and $f^y \in L^1(\mathbb{R})$ for a.e. $y$ but

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_x(y)\, dy\, dx \neq \int_{\mathbb{R}} \int_{\mathbb{R}} f^y(x)\, dx\, dy \qquad (1)$$

b. What does Fubini's theorem assert about such $f$ ($f$ that satisfy (1))?

c. What does Tonelli's theorem assert about such $f$ ($f$ that satisfy (1))?

**Question 3.** Prove that a normed vector space is a Banach space if and only if every absolutely convergent series is convergent. As part of your answer, state the definitions of "Banach space," "absolutely convergent" and "convergent."

**Question 4.** Denote by A the smallest algebra of subsets of R that contains all bounded intervals. Denote by A the collection of countable unions of sets in A. Denote by $^1$ the outer measure on the power set P(R) induced by the premeasure on A that assigns to any bounded interval its Euclidean length, and to any unbounded interval 1 .

a. Let E R. What does \E is $^1$ -measurable" (i.e. outer measurable) mean?

b. How is the collection of $^1$-measurable sets related to the collection of $^1$ -measurable sets?

c. Prove that for any E R and any > 0, there exists A 2 A with E A and $^1$ (A) $^1$ (E) + .