ANALYSIS QUALIFYING EXAM

SEPTEMBER , 2012

REAL ANALYSIS

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Question 1 (30 points)
Let $(X; M; \ )$ be a measure space. A measure M , with $(E) = \infty$, there
$\in M$ such that $0 < \ (F) < \infty$.
is semi nite and $(E) = \infty$, for any $C > 0$ there exists an $F \in M$ such that $C < \ (F) <$

(20 points)
) be a measure space and $L^p(X) = \{f : X \to \mathbb{C} : f$ is measurable and $\|f\|_p < \infty\}$.
$^p(X)$ is a Banach space for $1 \ p < \infty$ by proving
$^p(X)$ then $\|f + g\|_p \ \|f\|_p + \|g\|_p$
complete.

(30 points)
riation of a complex measure is the positive measure $| \ |$ determined by the property
$f d$ for some positive measure , $f \in L^1(\ )$, then $d| \ | = |f| d$.
is is well de ned by showing the following;
ays exists such a measure.
tion is independent of .

(20 points)
$:\|_2$ be two norms on a vector space V such that $\|v\|_1 \ \|v\|_2$ for all $v \in V$. If V is complete
to both norms, prove that they are equivalent.
e Banach spaces and let $T_n \in L(X; Y)$ such that $T(x) = \lim_{n \to \infty} T_n(x)$ exists for all $x \in X$.
$\in L(X; Y)$.

COMPLEX ANALYSIS

You should attempt all the problems. Partial credit will be give for serious efforts

(1) Compute the following integral:

$$\int_0^\infty \frac{\log x}{x^2+1}\,dx$$

(2) Let $\mathbb{A} = \{a_0, a_1, \ldots a_n\}$ be a finite set of (distinct) points in the unit disk D. Define

$$B_{\mathbb{A}}(z) = \Pi^{n}_{=0} \frac{z-a}{1-\bar{a}\,z}\frac{|a|}{a}, \qquad \text{for } z \; \square \; D$$

where if $a = 0$, we set $\frac{|\,_i|}{_i} = 1$.

(a) Prove that $B(z)$ maps D to D and maps the unit circle to the unit circle.

(b) Let $T : D \to D$ be a fractional linear transformation that maps the unit disk onto itself. Prove that

$$B_{\mathbb{A}} \circ T = \lambda B_{\;-1(\mathbb{A})}$$

where λ is a constant with $|\lambda| = 1$ and $T^{-1}(\mathbb{A}) = \{T^{-1}(a_0), \ldots, T^{-1}(a_n)\}$.

(c) Let $f : D \to D$ be an analytic function with $f(a\;) = 0$ for each a □ $\mathbb{A}$. Prove that $|f(z)| \leq |B_{\mathbb{A}}(z)|$ for all z □ D.

(3) The expression

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is called the c z n o f. If $f(z)$ has a zero or pole of order m $(m > 1)$ at z_0, show that $\{f, z\}$ has a pole at z_0 of order 2 and calculate the coe□cient of $\frac{1}{(\;-\;_0)^2}$ in the Laurent development of $\{f, z\}$.

(4) Let f be a bounded

(a) Show that the area integral

$$\iint_{|\ |<1} \frac{f(z)\,dx\,dy}{(1-\bar{z}\zeta)^2}, \qquad z = x + yi$$

is equal to

$$\int_0^1 \left(\int_{|\ |=1} \frac{z f(rz)}{i(z - r\zeta)^2}\, dz \right) r\,dr$$

(Hint: use polar coordinates)

(b) Use part (a) to prove

$$f(\zeta) = \frac{1}{\pi} \iint_{|\ |<1} \frac{f(z)\,dx\,dy}{(1-\bar{z}\zeta)^2}$$