Real Analysis Qual

June 10, 2016

Problem 1. Let be the equivalence relation on the interval $[0,1]$ given by $x \quad y$ i□ $x - y \quad \mathbb{Q}$. Choose one element from each equivalence class (using the Axiom of Choice). Let $A \quad [0,1]$ denote the set of these chosen elements. For a given set $B \quad \mathbb{R}$, define $B + x = \{y + x \mid y \quad B\}$.

(a) Show that the sets $A + q$, for $q \quad \mathbb{Q} \cap [-1,1]$, are disjoint.

(b) Show that $[0,1] \quad \bigcup_{q \quad \mathbb{Q}\cap[-1,1]}(A + q) \quad [-1,2]$.

(c) Show that A

(c) Prove Fatou's Lemma, which states that

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$$

for any sequence (f_n) of non-negative, measurable functions. To get started, let $g_n = \inf_{i \ge n} f_i$. Observe th A with

$\mu(X) < \infty$. Fix $E_1, \ldots, E_n \quad A$ and $c_1, \ldots, c_n \quad \mathbb{R}_{\ge 0}$. Define $\nu : A \to [0,\infty]$ by

$$\nu(A) = \sum_{i=1}^{n} c_i \mu(A \cap E_i).$$

(a) Show that ν is a measure.

(b) Show that ν is absolutely continuous with respect to μ.

Complex Analysis Qualifying Exam – Spring 2016

Please answer the following problems. Explain your argument carefully – if you refer to a well-known theorem from class, please state the theorem precisely and explain why it applies.

Notation: D = open unit disk, C = C − $\{0\}$, H = upper half plane.

1) Find a holomorphic function f on C such that

   - f is a pointwise limit of polynomial functions, but
   - f is not a uniform limit of polynomial functions (that is, there is no sequence of polynomials that converges to f uniformly on compact subsets of C).

   Prove both assertions for your choice of f.

2) Find a biholomorphism between H and the region

   $$U = \{z \quad \mathrm{C}\, | |z-1| < 1, |z-i| < 1\}.$$

   It is enough to write down explicitly functions whose composition yields a biholomorphism from H to U or from U to H.

3) Let U ▫ C be a simply connected region. For any point a U, the Green function of U

where $\wp(z)$ is the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Justify carefully any techniques you use.